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Методичні вказівки до практичних занять з дисципліни "Вища математика" за розділом "Невизначений інтеграл" за освітнім рівнем "Бакалавр" для іноземних студентів усіх факультетів (англійською мовою)

Methodical instructions for practical classes on the discipline "Higher Mathematics" under the section "Indefinite integral" by educational level "Bachelor" for foreign students of all faculties (in English)

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CONTENT

INDEFINITE INTEGRAL

1. DEFINITIONS AND SOME PROPERTIES. TABLE OF INTEGRALS

Function *F(x)* is called antiderivative function for function *f(x)* at a certain interval, if at each point of this interval runs equality

$$\frac{dF(x)}{dx} = f(x).$$

For example function *sin x* is antiderivative function or the function $y = \cos x$ at the interval $(-\infty,\infty)$, because $(\sin x)' = \cos x$.

Note: if *F(x)* is antiderivative function for function $f(x)$, then expression $F(x)+C$, where C is arbitrary constant, is antiderivative function for function $f(x)$ also, because runs equality

$$\frac{d}{dx}(F(x)+C) = f(x).$$

It's not hard to prove that the expression $F(x)+C$ exhausts all antiderivative functions for $f(x)$.

The set of all antiderivative functions for $f(x)$ is called an indefinite integral of a function $f(x)$ and is indicated by the symbol $\int f(x)dx = F(x)+C$, if $F'(x) = f(x)$.

The finding of antiderivative function for $f(x)$ is called the integration of the function. Operation of function integration is reverse to the differentiation operation. Integration is a more complex task than differentiating a function.

From the definition of integral and derivatives we have a table of integrals for the basic functions in the form:

1. $\int x^2 dx = \frac{x^{\alpha+1}}{\alpha+1} + C$
2. $\int \frac{dx}{x} = \ln|x| + C$
3. $\int e^x dx = e^x + C$
4. $\int a^x dx = \frac{a^x}{\ln a} + C$
5. $\int \sin x dx = -\cos x + C$
6. $\int \cos x dx = \sin x + C$
7. $\int \frac{dx}{\cos^2 x} = tgx + C$
8. $\int \frac{dx}{\sin^2 x} = -ctgx + C$
9. $\int tgx dx = -\ln|\cos x| + C$
10. $\int ctgx dx = \ln|\sin x| + C$
11. $\int \frac{dx}{a^2 + x^2} = \frac{1}{2} arctg \frac{x}{a} + C$
12. $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \ln\left|\frac{x-a}{x+a}\right| + C$
13. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin \frac{x}{a} + C$
14. $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln\left|x + \sqrt{x^2 \pm a^2}\right| + C$
15. $\int \frac{dx}{\sin x} = \ln\left|tg \frac{x}{2}\right| + C$
16. $\int \frac{dx}{\cos x} = \ln\left|tg\left(\frac{\pi}{4} - \frac{x}{2}\right)\right| + C$

In the table of derivatives formulas (9)-(16) are absent, but their validity is easy to verify by differentiation.

We note some properties of an indefinite integral. Let for $x\in(a,b)$ runs $\int f(x)dx = F(x)+C$, $\int \varphi(x)dx = \phi(x)+C$.

Since $\frac{dF(x)}{dx} = f(x)$, …….. $\frac{d\phi(x)}{dx} = \varphi(x)$,
$\frac{d(F(x)\pm\phi(x))}{dx} = f(x)+\varphi(x)$, and $\frac{d(kF(x))}{dx} = kf(x)$, *(k-const),*

1. $(\int f(x)dx)' = f(x)$, 2. $d(\int f(x)dx) = f(x)dx$,

3. $\int dF(x) = F(x)+C$, 4. $\int kf(x)dx = k\int f(x)dx$,

5. $\int (f(x)\pm\varphi(x))dx = \int f(x)dx \pm \int \varphi(x)dx$.

Thus, the integral of the algebraic sum is equal to the algebraic sum of integrals, a constant factor can be taken out from the integral sign.

Example 1.

$$\int \frac{1-2x^2 tgx + 4\sqrt[3]{x}}{x^2}dx = \int \frac{1}{x^2}dx + \int \frac{-2x^2 tgxdx}{x^2} + \int 4\frac{\sqrt[3]{x}}{x^2}dx =$$

$$= \int x^{-2}dx - 2\int tgxdx + 4\int x^{-\frac{5}{3}}dx = \frac{x^{-2+1}}{-2+1} - 2(-\ln|\cos x|) + 4\frac{x^{-\frac{5}{3}+1}}{-\frac{5}{3}+1} =$$

$$= -\frac{1}{x} + 2\ln|\cos x| - \frac{6}{\sqrt[3]{x^2}} + C$$

The integral is found using the properties 4 and 5 and formulas (1) and (9) of the integrals table.

Example 2. $\int \frac{dx}{\sqrt{4-25x^2}} = \int \frac{dx}{5\sqrt{\frac{4}{25}-x^2}} = \frac{1}{5}\arcsin\frac{5x}{2} + C$.

The integral is reduced to tabulated one (formula (13)) with a value $a^2 = \frac{4}{25} \Rightarrow a = \frac{2}{5}$.

2. INTEGRATION USING SUBSTITUTION

There are several methods for finding the antiderivative function. In particular, it is a method of integration using substitution or replacement of a variable. Let in *(a,b)* runs the equality

$$\int f(x)dx = F(x) + C. \qquad (2.1)$$

Assume that the function $x = \varphi(t)$ is continuously differentiated in the interval (α, β), and let $\alpha < \varphi(t) < b$ for all $t \in (\alpha, \beta)$

The derivative of the composite function $F(\varphi(t))$ is defined in the form

$$\frac{dF[\varphi(t)]}{dt} = F'(\varphi(t))\varphi'(t).$$

Since $F'(x) = f(x)$, then $\frac{d}{dt}F[\varphi(t)] = f(\varphi(t))\varphi'(t)$, whence

$$\int f[\varphi(t)]\varphi'(t)dt = F(\varphi(t)) + C. \qquad (2.2)$$

We note that for (1) and (2)

$$\int f(x)dx = \int f[\varphi(t)]\varphi'(t)dt \text{ for } x = \varphi(t). \qquad (2.3)$$

We obtained this formula formally, having accepted that $x = \varphi(t), dx = \varphi'(t)dt$.

Remark. By rearranging the letters *x* and *t* in formula (2.3), we have

$$\int f[\varphi(x)]\varphi'(x)dx = \int f(t)dt \text{ для } t = \varphi(x). \qquad (2.4)$$

Formally, we get this formula by accepting $t = \varphi(x), dt = \varphi'(x)dx$.

<u>Example 3</u>. Find integral $\int (ax+b)^n dx, n \neq -1, a \neq 0$.

Assume $ax + b = t, x = \frac{t-b}{a}, dx = \frac{1}{a}dt$.

Applying formula (2.3), we obtain

$$\int (ax+b)^n dx = \int t^n \frac{dt}{a} = \frac{1}{a}\frac{t^{n+1}}{n+1} + C = \frac{1}{a}\frac{(ax+b)^{n+1}}{n+1} + C.$$

<u>Example 4</u>. Find integral $\int xctg(x^2+3)dx$.

$$\int xctg(x^2+3)dx = \begin{vmatrix} t = x^2 + 3 \\ dt = 2xdx \\ xdx = \frac{1}{2}dt \end{vmatrix} = \int ctgt \cdot \frac{1}{2}dt = \frac{1}{2}\ln|\sin t| + C = \frac{1}{2}\ln|\sin(x^2+3)| + C.$$

In this case, we used the substitution $t = x^2 + 3$.

Example 5.

$$\int \frac{f'(x)}{f(x)}dx = \begin{vmatrix} t = f(x) \\ dt = f'(x)dx \end{vmatrix} = \int \frac{dt}{t} = \ln|t| + C = \ln|f(x)| + C.$$

Thus, if the derivative of the denominator is equal to a numerator, then the integral is equal to the logarithm of the module of the denominator.

3. INTEGRATION BY PARTS

Let u and v be continuously differentiable functions of the variable x in the interval (a, b). Then:

$d(uv) = vdu + udv$, так що $ud'v = d(uv) - v(du)$.

By getting indefinite integrals from both parts and taking into account what $\int d(u,v) = uv + C$, we have

$$\int udv = uv - \int vdu. \quad (3.5)$$

Formula (3.5) makes it possible to lead the finding of integral $\int udv$ to finding the integral $\int vdu$, which may be taken more easily.

For example, integrals of the form

$$\int P(x)e^{ax}dx,\ \int P(x)\cos axdx,\ \int P(x)\sin axdx, \quad (3.6)$$

$$\int P(x)\ln xdx,\ \int P(x)arctgxdx,\ \int P(x)\arcsin xdx. \quad (3.7)$$

where *P(x)* – the polynomial, should be integrated into parts, taking in the case (3.6) $u = P(x)$, and in the case (3.7) – $dv = P(x)dx$.

Example 6.

$$\int x^2 \cos xdx = \begin{vmatrix} u = x^2 & du = 2xdx \\ dv = \cos xdx & v = \int \cos xdx = \sin x \end{vmatrix} =$$

$$= x^2 \sin x - 2\int x \sin xdx = \begin{vmatrix} u = x & du = dx \\ dv = \sin xdx & v = -\cos x \end{vmatrix} =$$

$$= x^2 \sin x - 2(-x\cos x - \int(-\cos x)dx) = x^2 \sin x + 2x\cos x - 2\sin x + C$$

In this case, the formula of integration by parts was applied twice.

Example 7.

$$\int \arccos 5xdx = \begin{vmatrix} u = \arccos 5x & du = -\dfrac{5dx}{\sqrt{1-25x^2}} \\ dv = dx & v = x \end{vmatrix} =$$

$$= xar\cos 5x + \int \frac{5xdx}{\sqrt{1-25x^2}} = \begin{vmatrix} 1-25x^2 = t \\ -50xdx = dt \end{vmatrix} =$$

$$= x\arccos 5x - \frac{1}{10}\int \frac{dt}{\sqrt{t}} = \arccos 5x - \frac{1}{5}\sqrt{t} + C = x\arccos 5x - \frac{1}{25}\sqrt{1-25x^2} + C$$

Example 8.

$$I = \int e^{2x}\sin xdx = \begin{vmatrix} u = e^{2x} & du = 2e^{2x}dx \\ dv = \sin dx & v = -\cos x \end{vmatrix} = -e^{2x}\cos x + 2\int e^{2x}\cos xdx =$$

$$= \begin{vmatrix} u = e^{2x} & du = 2e^{2x}dx \\ dv = \cos xdx & v = \sin x \end{vmatrix} = -e^{2x}\cos x + 2(e^{2x}\sin x - 2\int e^{2x}\sin xdx)$$

Since we returned to the initial integral, we got the equation in relation to it in form

$$\mathrm{I} = 2\mathrm{e}^{2x}\sin x - \mathrm{e}^{2x}\cos x - 4\mathrm{I},$$

whence

$$\mathrm{I} = \frac{2\mathrm{e}^{2x}\sin x - \mathrm{e}^{2x}\cos x}{5} + \mathrm{C}.$$

4. INTEGRATION OF RATIONAL FUNCTIONS

In algebra it is proved that every polynomial of form

$$Q(x) = a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n$$

can be represented in the form

$$Q(x) = a_0(x-\alpha)^r(x-\beta)^s \ldots (x^2+px+q)^k, r+s+\ldots+zk = n, \qquad (4.8)$$

where $\alpha, \beta, \ldots$ are real roots of equality $Q(x) = 0$.

In this expression all values $\alpha_0, \alpha, \beta, \ldots, P, q$ are real, and polynomials x^2+px+q have no real roots. Expression (4.8) is called decomposition of decomposition of a polynomial on elementary factors.

A rational function (fraction) is a function defined in the form of a fraction of two polynomials at those points in which the denominator does not vanish. Consequently, a rational function may be presented in the form of a fraction $P(x)/Q(x)$, where *P(x)* and *Q(x)* are polynomials.

A fraction is called ***irreducible*** if the polynomials *P(x)* and *Q(x)* have no common elementary factors.

A fraction is said to be ***correct*** if the degree of the polynomial *P(x)* is lower than the degree of the polynomial *Q(x)*.

If the degree of the numerator is greater than or equal to the degree of the denominator, then we have after dividing:

$$\frac{P(x)}{Q(x)} = W(x) + \frac{R(x)}{Q(x)},$$

where *W(x)* is certain polynomial, and *R(x)* is polynomial of degree lower than *Q(x)*.

Example 9. $\frac{x^3+x+1}{x^2+1}=x+\frac{1}{x^2+1}$.

Theorem. Correct irreducible fraction $\frac{P(x)}{Q(x)}$, where

$$Q(x)=a_0(x-\alpha)^n(x-\beta)^s\ldots(x^2+px+q)^k$$

can be represented in form

$$\frac{P(x)}{Q(x)}=\frac{A}{(x-\alpha)^2}+\frac{B}{(x-\alpha)^{r-1}}+\ldots+\frac{C}{(x-\alpha)}+\frac{D}{(x-\beta)^s}+\frac{\varepsilon}{(x-\beta)^{s-1}}+\ldots+$$
$$+\frac{F}{x-\beta}+\ldots+\frac{Gx+H}{(x^2+px+q)^k}+\frac{Jx+K}{(x^2+px+q)^{k-1}}+\ldots+\frac{Lx+M}{x^2+px+q}. \qquad (4.9)$$

In this decomposition, *A, B, C,..., L, M* are constant numbers. Decomposition (4.9) is called the decomposition of a rational function on elementary (simplest) fractions. Equality (4.9) is applicable to all valid *x*, except for the values of the real roots of the equation $Q(x)=0$, which are $x=\alpha,\beta,\ldots$. Each polynomial' $Q(x)$ multiplier is part of the denominator in development (4.9) in all degrees, starting with the degree that it has in decomposition (4.8), and ending with the first degree.

The numerators of the fractions included in the decomposition (4.9) are either constant numbers or polynomials of the first degree, depending on whether the denominator is a certain degree of the polynomial of the first or second degree.

To determine the numbers *A, B, C,...* we multiply the two parts of the relation (4.9) by $Q(x)$. Free oneself us in this way from the denominators, we will represent a polynomial, obtained in the right side, in powers of *x*. Since the equality between a polynomial $P(x)$ and the polynomial, which will be on the right side, holds for all values of *x*, then the coefficients standing at equal powers of the variable *x* are equal. We obtain a system of *n* equations of the first degree with *n* unknowns *A, B, C,....* Having solved the received system, we will define numbers *A, B, C,....*

Example 10. Decompose a rational function for the simplest fractions

$$\frac{3x-3}{x^2-x-2}.$$

Since $x^2-x-2=(x-2)(x+1)$, we assume that

$$\frac{3x-3}{x^2-x-2}=\frac{A}{x-2}+\frac{B}{x+1},$$

where, multiplying both parts by x^2-x-2 we get

$$3x-3=A(x+1)+B(x+2),$$

so
$$3x-3=(A+B)x+A-2B.$$

So $A+B=3$, and $A-2B=-3$, whence A=1, B=2.

Thus $\frac{3x-3}{x^2-x-2}=\frac{1}{x-2}+\frac{2}{x+1}$.

Example 11. Decompose a rational function for the simplest fractions

$$\frac{4x^2+4x+6}{x(x+1)(x-2)}.$$

We apply a different method that quickly leads to a goal, when the denominator has only valid roots.

Assume $\frac{4x^2+4x+6}{x(x+1)(x-2)}=\frac{A}{x}+\frac{B}{x+1}+\frac{C}{x-2}$, whence

$4x^2+4x+6=A(x+1(x-2)+Bx(x-2)+Cx(x+1)$.

Having taken turns x = 0; 2; -1, we have

$$6 = -2A,\ 30 = 6C,\ 6 = 3B.$$

So $A=-3; B=2; C=5$ and $\frac{4x^2+4x+6}{x(x+1)(x-2)}=-\frac{3}{x}+\frac{2}{x+1}+\frac{5}{x-2}$.

Example 12. Decompose a rational function for the simplest fractions

$$\frac{x^2+2x-1}{(x-1)(x^2+1)}=\frac{A}{x-1}+\frac{Bx+C}{x^2+1}.$$

We get $x^2+2x-1=A(x^2+1)+(Bx+C)(x-1)$.

Assuming x=1, we have: 2=2A, A=1. Multiplying and comparing coefficients at the same degrees x, we find

$$1 = A + B,\ 2 = -B + C,\ -1 = A - C,$$

so $B=0, C=2$.

Example 13. Decompose a rational function for the simplest fractions

$$\frac{3x^2+2x^2+1}{x^2(x^2+1)}=\frac{A}{x^2}+\frac{B}{x}+\frac{Cx+D}{x^2+1}.$$

Having freed from the denominators and comparing the coefficients, we have

$$B + C = 3,\ A + D = 2,\ B = 0,\ A = 1,$$

thus

$$A = 1,\ B = 0,\ C = 3,\ D = 1.$$

After decomposing the rational function to elementary fractions, we reduce the integral from rational functions to the integrals of the following types

1. $\int\frac{Adx}{x-\alpha}=A\ln|x-\alpha|+C$, 2. $\int\frac{Adx}{(x-\alpha)^2}=-\frac{A}{(r-1)(x-\alpha)^{r-1}}+C$,

3. $\int \frac{Ax+B}{x^2+px+q}dx$, 4. $\int \frac{Ax+B}{(x^2+px+q)^r}, r>1$,

and in polynomials of type 3 and 4 the polynomial x^2+px+q has no real roots.

To find integrals of type 3 and 4 we introduce a new variable $z=x+\frac{P}{2}$, where we find $x^2+px+q=\left(x+\frac{p}{x}\right)^2+q-\frac{p^2}{4}=z^2+a^2$, де $a^2=q-\frac{p^2}{4}$.

So we get

$$\int \frac{Ax+B}{x^2+px+q}dx=\left|\begin{matrix} z=x+\frac{p}{2} \\ dz=dx \end{matrix}\right|=\int \frac{Az+B-A\frac{p}{2}}{z^2+a^2}dz=$$

$$=\frac{A}{2}\int \frac{2zdz}{z^2+a^2}+\left(B-A\cdot\frac{p}{2}\right)\int \frac{dz}{z^2+a^2}=\frac{A}{2}ln\left|z^2+a^2\right|+\frac{B-A\frac{p}{2}}{a}arctg\frac{z}{a}+C=$$

$$=\frac{A}{2}ln\left|x^2+px+q\right|+\frac{B-A\frac{p}{2}}{a}arctg\frac{x+\frac{p}{2}}{a}+C;$$

$$\int \frac{Ax+B}{(x^2+px+q)^2}dx=A\int \frac{zdz}{(z^2+a^2)^2}+\left(B-A\frac{p}{2}\right)\int \frac{dz}{(z^2+a^2)^2}.$$

We find the first integral by replacing $z^2+a^2=t$

$$\int \frac{(x+1)dx}{x^3-2x^2-x+2}$$

We find another integral using the reduction formula

$$\int \frac{dz}{(z^2+a^2)^r}=-\frac{1}{a^2}\left(\frac{z}{2(r-1)(z^2+a^2)^{r-1}}+\frac{2r-3}{2r-2}\int \frac{dz}{(z^2+a^2)^{r-1}}\right).$$

<u>Example 14</u>. Find $\int \frac{(3x-3)dx}{x^2-x-2}$.

Since $\frac{3x-3}{x^2-x-2}=\frac{1}{x-2}+\frac{2}{x+1}$ (see example 10), then

$$\int \frac{3x-3}{x^2-x-2}dx=\int \frac{dx}{x-2}+2\int \frac{dx}{x+1}=\ln|x-2|+2\ln|x+1|+C.$$

<u>Example 15</u>. Find $\int \frac{3x^3+2x^2+1}{x^2(x^2+1)}dx$.

Since $\frac{3x^3+2x^2+1}{x^2(x^2+1)}=\frac{1}{x^2}+\frac{3x+1}{x^2+1}$ (see example 13), then

$$\int \frac{3x^3+2x^2+1}{x^2(x^2+1)}dx=\int \frac{dx}{x^2}+3\int \frac{xdx}{x^2+1}+\int \frac{dx}{x^2+1}=-\frac{1}{x}+\frac{3}{2}\ln\left|x^2+1\right|+arctgx+C.$$

5. INTEGRATION OF TRIGONOMETRIC FUNCTIONS

If $R(u,v)$ is rational function of variables *u* and *v*, then the integral $\int R(\sin x,\cos x)dx$ can be reduced to an integral from rational function using substitution function $tg\frac{x}{2}=t$.

So we get

$$\sin x=\frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}}=\frac{2tg\frac{x}{2}}{1+tg^2\frac{x}{2}}=\frac{2t}{1+t^2},$$

$$\cos x=\frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}}=\frac{1-tg^2\frac{x}{2}}{1+tg^2\frac{x}{2}}=\frac{1-t^2}{1+t^2}.$$

Since $x=2arctgt$, then $dx=\frac{2dt}{1+t^2}$.

Since $\sin x,\cos x$ and dx are expressed rationally through the variable *t*, then the integral $\int R(\sin x,\cos x)dx$ with the help of the specified substitution goes to the integral of the rational function. Substitution $tg\frac{x}{2}=t$ is called a universal trigonometric substitution.

<u>Example 16</u>. Find $\int\frac{dx}{\sin x}$.

Apply a universal trigonometric substitution:

$$\int\frac{dx}{\sin x}=\left|\begin{matrix} tg\frac{x}{2}=t, dx=\frac{2dt}{1+t^2} \\ \sin x=\frac{2t}{1+t^2}\end{matrix}\right|=\int\frac{\frac{2dt}{1+t^2}}{2t/(1+t^2)}=\int\frac{dt}{t}=\ln|t|+C=\ln\left|tg\frac{t}{2}\right|+C.$$

In some cases, the integral $R(\sin x,\cos x)dx$ can be found faster by using the following substitutions:

a) if $R(u,v)$ is an odd function of the variable *u*, that is if $R(-u,v)=-R(u,v)$, then the substitution is applied $\cos x=t$;

б) if $R(u,v)$ is an odd function of the variable *v*, then the substitution can be applied $\sin x=t$;

в) if $R(u,v)$ is an even function of the variables *u* та *v* , that is if $R(-u,-v)=R(u,v)$, then the substitution is applied $tgx=t$.

Example 17. Find $\int \frac{\cos^3 x}{\sin^2 x+1}dx$.

The integral function is odd relative $\cos x$, so

$$\int \frac{\cos^3 xdx}{\sin^2 x+1} = \left| \begin{matrix} \sin x = t \\ \cos xdx = dt \end{matrix} \right| = \int \frac{1-t^2}{1+t^2}dt = \int \left(-1 + \frac{2}{1+t^2} \right) dt = -t + 2arctgt + C =$$

$$= -\sin x + 2arctg(\sin x) + C.$$

Example 18. Find $\int \frac{dx}{\sin^2 x + 4\sin x\cos x + 5\cos^2 x}$.

Since the integral function is even relative $\sin x$ and $\cos x$, so

$$\int \frac{dx}{\sin^2 x + 4\sin x\cos x + 5\cos^2 x} = \left| \begin{matrix} tgx = t \\ \frac{dx}{\cos^2 x} = dt \end{matrix} \right| =$$

$$= \int \frac{dt}{t^2+4t+5} = \int \frac{dt}{(t+2)^2+1} = arctg(t+2) + C =$$

$$= arctg(tgx+2) + C.$$

6. INTEGRATION OF CERTAIN IRRATIONAL FUNCTIONS

If $R(x,y,...,u)$ is rational function of variables $x,y,...,u$, where

$$y = \left(\frac{ax+b}{cx+d} \right), ..., u = \left(\frac{ax+b}{cx+d} \right)^{\frac{r}{5}} \quad (\mathrm{m,n,...,r,s-integer}),$$

then $\int R(x,y,...,u)dx$ can be got as rational function using the substitution

$$\frac{ax+b}{cx+d} + t^p,$$

where p is the total smallest denominator of all degrees of power.

Example 19.

$$\int \frac{dx}{(\sqrt[3]{x}-1)\sqrt{x}} = \left| \begin{matrix} x = t^6 \\ dx = 6t^5dt \end{matrix} \right| = \int \frac{6t^5dt}{(t^2-1)t^3} = \int \frac{6t^2dt}{t^2-1} = \int \left(6 + \frac{6}{t^2-1} \right) dt =$$

$$= 6t + 3ln\left| \frac{t-1}{t+1} \right| + C = 6 \cdot \sqrt[6]{x} + 3ln\left| \frac{\sqrt[6]{x}-1}{\sqrt[6]{x}-1} \right| + C.$$

Example 20.

$$\int \frac{1}{x}\sqrt{\frac{1-x}{x}}dx = \left| \begin{matrix} \frac{1-x}{x} = t^2, \\ x = \frac{1}{t^2+1}, dx = \frac{-2tdt}{(t^2+1)^2} \end{matrix} \right| = \int (t^2+1)\cdot t \cdot \frac{-2tdt}{(t^2+1)^2} =$$

$$= \int \frac{-2t^2dt}{t^2+1} = \int \left(-2 + \frac{2}{t^2+1}\right)dt =$$

$$= -2t + 2arctgt + C = -\sqrt{\frac{1-x}{x}} + 2arctg\sqrt{\frac{1-x}{x}} + C.$$

Integral in form $\int R(x,\sqrt{ax^2+bx+c})dx$, where $R(x,y)$ is rational function, after substitution $z = x + \frac{b}{2a}$ reduced to one of following types:

1. $\int R\left(z,\sqrt{m^2-z^2}\right)dz$ 2. $\int R\left(z,\sqrt{z^2-m^2}\right)dz$;
3. $\int R\left(z,\sqrt{z^2+m^2}\right)dz$

The integral of type 1 is rationalized by substitution $z = m\sin t$, the integral of type 2 – by substitution $z = \frac{m}{\cos t}$, the integral of type 3. – by substitution $z = mtgt$.

Example 21.

$$\int \frac{x^2dx}{\sqrt{256-25x^2}} = \frac{1}{5}\int \frac{x^2dx}{\sqrt{\frac{256}{25}-x^2}} = \left| \begin{matrix} x = \frac{16}{5}\sin t \\ dx = \frac{16}{5}\cos tdt \end{matrix} \right| = \frac{1}{5}\cdot\int \frac{\left(\frac{16}{5}\right)^2 \sin^2 t \frac{16}{5}\cos tdt}{\sqrt{\frac{256}{25}-\frac{256}{25}\sin^2 t}} =$$

$$= \frac{256}{125}\cdot\int \sin^2 tdt = \frac{128}{125}\cdot\int(1-\cos 2t)dt = \frac{128}{125}\cdot\left(t-\frac{1}{2}\sin 2t\right)+C =$$

$$= \frac{128}{125}\cdot(t-\sin t\cos t)dt + C = \frac{128}{125}\cdot\arcsin\frac{5x}{16} - \frac{128}{125}\cdot\frac{5}{16}x\cdot\sqrt{1-\left(\frac{5x}{16}\right)^2}+C =$$

$$= \frac{128}{125}\cdot\arcsin\frac{5x}{16} - \frac{1}{50}x\cdot\sqrt{256-25x^2}+C.$$

$$\int\frac{x^3dx}{\sqrt{25+x^2}}=\left|\begin{matrix}x=5tdt\\ dx=\dfrac{5dt}{\cos^2 t}\end{matrix}\right|=\int\frac{(5tdt)^3 5dt}{\sqrt{25+25tg^2 t}\cos^2 t}=125\int\frac{\sin^3 tdt}{\cos^4 t}=$$

$$=\left|\begin{matrix}\cos t=z\\ -\sin tdt=dz\end{matrix}\right|=125\int\frac{(1-z^2)(-dz)}{z^4}=125\int\frac{z^2-1}{z^4}dz=125\int(z^{-2}-z^{-4})dz=$$

$$=125\left(\frac{z^{-1}}{-1}+\frac{z^{-3}}{-3}\right)+C=125\left(\frac{1}{3\cos^3 t}-\frac{1}{\cos t}\right)+C.$$

Since $\cos t=\dfrac{1}{\sqrt{1+tg^2 t}}=\dfrac{1}{\sqrt{1+\left(\dfrac{x}{5}\right)^2}}=\dfrac{5}{\sqrt{25+x^2}}$, then

$$I=\frac{125}{3\cos t}\left(\frac{1}{\cos^2 t}-3\right)+C=\frac{25}{3}\sqrt{25+x^2}\left(\frac{25+x^2}{25}-3\right)+C=$$

$$=\frac{1}{3}\sqrt{25+x^2}(x^2-50)+C.$$

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